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Geometric Character of Black-Hole Entropy

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The geometry of the neighborhood near an event horizon is similar to the Rindler metric, which leads to the thermal effect of black holes. The entropy of the scalar field and the Dirac field are calculated in the black-hole background. The entropy of the scalar field, which is proportional to the area of the event horizon, is naturally derived. Under the condition of large-mass black hole, the entropy of the Dirac field is still proportional to the area of the horizon. These results can be applied to a large class of black holes. A new method for calculating the black hole entropy is proposed which makes it easy to calculate the entropy of a high-spin field in the black-hole background. We also consider extreme black holes and point out that the topological entropy only has classical meaning.

1. INTRODUCTION

The dynamical origin of black-hole entropy has been an interesting and important problem in theoretical physics since the thermal radiation of black hole was discovered by Hawking (1975). To understand black-hole entropy we need a good theory of quantum gravity because of the statistical meaning of entropy. Valuable discussions in the semiclassical frame include the brick-wall model (t' Hooft, 1985) and entanglement entropy (Frolov and Novikov, 1993). The latter is associated with modes and correlations hidden from external observers by the presence of a horizon. There is correlation between the internal and external modes, and the entanglement entropy can be found by counting external models. Using the brick-wall model, much work has been devoted to studying the relation

$$S = \alpha A \tag{1}$$

for particular cases, such as the Schwarzchild black hole (SBH) or the

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Reisner–Nordstrom black hole (RN BH). The above formula with its geometric character may be universal and intrinsic, independent of the metric of the black hole, but the calculation of black-hole entropy depends on the particular case. Therefore, it is significant that the black-hole entropy is given by a universal expression for the general case.

It is well known that the existence of a horizon is important for the thermal effect of a black hole. Hawking radiation derives from vacuum fluctuation near the event horizon. The black-hole entropy is related to the horizon. The central idea of this paper is that the black-hole entropy derives from the contribution of matter fields near the horizon.

The following sections examine the geometry near the horizon of a general static, spherical black hole. In this background, the entropies of a scalar field and a Dirac field are calculated. The result that entropy is proportional to the area of horizon is naturally obtained. In particular, the calculation for a Dirac field becomes simple by using this new method.

2. GEOMETRY NEAR THE HORIZON OF A STATIC, SPHERICAL BLACK HOLE

For a general static, spherical black hole, the geometry of space-time is described by

$$ds^2 = -f(r)\, dt^2 + f^{-1}(r)\, dr^2 + R^2(r)\, d\Omega^2 \tag{2}$$

This equation describes a large class of black holes, including the Schwarzschild, Reissner–Nordstrom, and corresponding deSitter black holes, with cosmological constant or dilaton. The coordinate of the event horizon r_0 is determined by the equation $f(r_0) = 0$. In the near neighborhood of the horizon, the function $f(r)$ is expanded in a Taylor series

$$f(r) = a(r - r_0) \tag{3}$$

where $a = f'(r_0) = 2\kappa$, and κ is the surface gravity. Substituting (3) into (2), we can rewrite the metric near the horizon as

$$ds^2 = -a(r - r_0)\, dt^2 + \frac{dr^2}{a(r - r_0)} + R_0^2(r_0)\, d\Omega^2 \tag{4}$$

where $R_0^2(r_0) = R^2(r)\big|_{r_0}$. Although this looks unreasonable, we can regard it as an approximation after the entropy is calculated. We introduce the coordinate transformation

$$x = \frac{2}{\sqrt{a}} \sqrt{r - r_0} \tag{5}$$

Equation (4) becomes

$$ds^2 = -x^2\, dt'^2 + dx^2 + R_0^2\, d\Omega^2 \tag{6}$$

where $t' = \frac{1}{2}at = \kappa t$. Equation (6) is very similar to the Rindler metric. The Hawking temperature reads $T_H = 1/(2\pi)$ corresponding Rindler time t' with imaginary period $\beta = 2\pi$. It is easy to discuss the quantum matter fields in the background and calculate their entropy because of the simplicity of equation (6). This result is valid for a large class of black hole, so we have a new method for calculating the entropy.

Equation (6) seems to be well known (Zaslavskii, 1997).

3. BLACK-HOLE ENTROPY: SCALAR FIELD

The equation of a massless field in curved space reads

$$\frac{1}{\sqrt{-g}}\, \partial_\mu[\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi] = 0 \tag{7}$$

Its solution is supposed as

$$\Phi = e^{-i\omega t'} f(x) Y(\theta, \varphi) \tag{8}$$

Substituting it into equation (7), we get

$$x^{-2}\omega^2 + \frac{f''}{f} + \frac{f'}{xf} = \frac{-1}{R_0^2 Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, \frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = \lambda^2 R_0^{-2} \tag{9}$$

where

$$\lambda = \sqrt{l(l+1)}, \qquad l = 0, 1, 2, \ldots \tag{10}$$

is the separation constant. The equation for $f(x)$ reads

$$f'' + \frac{f'}{x} + \left(\frac{\omega^2}{x^2} - \frac{l(l+1)}{R_0^2}\right) f = 0 \tag{11}$$

Using the WKB approximation with $f(x) = \exp[iS(x)]$, we get

$$p_x = \partial_x S = \sqrt{\frac{\omega^2}{x^2} - k^2} \tag{12}$$

where $k^2 = l(l+1)/R_0^2$. According to the quasiperiodic condition, the mode number reads

$$n = \frac{1}{\pi} \int_{\epsilon}^{\omega/k} p_x dx = \frac{1}{\pi} \left[-\sqrt{\omega^2 - k^2\epsilon^2} + \omega \ln \frac{k\epsilon}{\omega - \sqrt{\omega^2 - k^2\epsilon^2}} \right] \quad (13)$$

ϵ is the ultraviolet cutoff. The number of modes with energy less than ω is given by

$$\begin{aligned} g(\omega) &= \int n(2l + 1)dl \\ &= \frac{2R_0^2}{\pi} \int k\, dk \left[\omega \ln\left(\frac{k\epsilon}{\omega - \sqrt{\omega^2 - k^2\epsilon^2}} \right) - \sqrt{\omega^2 - k^2\epsilon^2} \right] \end{aligned} \quad (14)$$

and the density of states is

$$\frac{dg(\omega)}{d\omega} = \frac{2R_0^2}{\pi} \int_0^{\omega/\epsilon} k\, dk \ln \frac{k\epsilon}{\omega - \sqrt{\omega^2 - k^2\epsilon^2}} = \frac{R_0^2\omega^2}{\pi\epsilon^2} \quad (15)$$

The free energy and entropy reduce to

$$\begin{aligned} F &= \frac{1}{\beta} \int_0^{\infty} dg(\omega) \ln(1 - e^{-\beta\omega}) \\ &= \frac{R_0^2}{\beta\pi\epsilon^2} \int_0^{\infty} d\omega\, \omega^2 \ln(1 - e^{-\beta\omega}) = -\frac{\pi^3 R_0^2}{45\beta^4\epsilon^2} \end{aligned} \quad (16)$$

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{4\pi^3 R_0^2}{45\beta^3\epsilon^2} \quad (17)$$

In the previous section we already pointed out the imaginary period is $\beta = 2\pi$; then

$$S = \frac{R_0^2}{90\epsilon^2} = \frac{A}{360\pi\epsilon^2} \quad (18)$$

where $A = \int \sqrt{g_{\theta\theta}g_{\varphi\varphi}}\, d\theta\, d\varphi = 4\pi R_0^2$ is the area of the horizon. Thus the geometric character of the black hole is obtained, which derives from the contribution of fields near the horizon.

4. BLACK-HOLE ENTROPY: DIRAC FIELD

Few papers are devoted to the problem of calculating the entropy of the Dirac field in curved space because of the nature of the complicated equation and the calculation. However, once we have successfully calculated the entropy of a scalar field, we can find the entropy of the Dirac field in a static, spherical background by a similar calculation as in the previous section. From the metric (6) the null tetrad is given by

$$l_\mu = \frac{1}{\sqrt{2}}(x, 1, 0, 0), \qquad n_\mu = \frac{1}{\sqrt{2}}(x, -1, 0, 0)$$
$$m_\mu = \frac{1}{\sqrt{2}}(0, 0, R_0, iR_0 \sin\theta), \qquad \overline{m}_\mu = \frac{1}{\sqrt{2}}(0, 0, R_0, -iR_0 \sin\theta) \tag{19}$$

According to Newman, and Penrose (1962), the nonzero spin coefficients are given by

$$\gamma = \epsilon = -\frac{1}{2\sqrt{2}x}, \qquad \alpha = -\beta = \frac{1}{2\sqrt{2}R_0}\,\mathrm{ctg}\,\theta \tag{20}$$

and the other differential operators are

$$D = l^\mu \partial_\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{x}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right), \qquad \nabla = n^\mu \partial_\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{x}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$$
$$\delta = m^\mu \partial_\mu = \frac{-1}{\sqrt{2}}\left(\frac{1}{R_0}\frac{\partial}{\partial\theta} + \frac{i}{R_0 \sin\theta}\frac{\partial}{\partial\varphi}\right)$$
$$\overline{\delta} = \overline{m}^\mu \partial_\mu = \frac{-1}{\sqrt{2}}\left(\frac{1}{R_0}\frac{\partial}{\partial\theta} - \frac{i}{R_0 \sin\theta}\frac{\partial}{\partial\varphi}\right) \tag{21}$$

Substituting (20) and (21) into the massless Dirac equation in curved space-time (Chandrasekhar, 1976), one has the following equations:

$$\left(\frac{1}{x}\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{1}{2x}\right)F_1 + \left(-\frac{1}{R_0}\frac{\partial}{\partial\theta} + \frac{i}{R_0 \sin\theta}\frac{\partial}{\partial\varphi} - \frac{1}{2R_0}\,\mathrm{ctg}\,\theta\right)F_2 = 0$$
$$\left(\frac{1}{x}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{1}{2x}\right)F_2 + \left(-\frac{1}{R_0}\frac{\partial}{\partial\theta} - \frac{i}{R_0 \sin\theta}\frac{\partial}{\partial\varphi} - \frac{1}{2R_0}\,\mathrm{ctg}\,\theta\right)F_1 = 0 \tag{22}$$
$$\left(\frac{1}{x}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{1}{2x}\right)G_1 + \left(\frac{1}{R_0}\frac{\partial}{\partial\theta} - \frac{i}{R_0 \sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{2R_0}\,\mathrm{ctg}\,\theta\right)G_2 = 0$$
$$\left(\frac{1}{x}\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{1}{2x}\right)G_2 + \left(\frac{1}{R_0}\frac{\partial}{\partial\theta} + \frac{i}{R_0 \sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{2R_0}\,\mathrm{ctg}\,\theta\right)G_1 = 0$$

The solutions are supposed as

$$F_1 = e^{-i\omega t}\frac{1}{\sqrt{x}} f_1(x)Y_1(\theta, \varphi)$$
$$F_2 = e^{-i\omega t}\frac{1}{\sqrt{x}} f_2(x)Y_2(\theta, \varphi) \tag{23}$$
$$G_1 = e^{-i\omega t}\frac{1}{\sqrt{x}} g_1(x)\tilde{Y}_1(\theta, \varphi)$$
$$G_2 = e^{-i\omega t}\frac{1}{\sqrt{x}} g_2(x)\tilde{Y}_2(\theta, \varphi)$$

Substituting into equation (22), we obtain

$$\begin{aligned} \hat{P}_- f_1(x) &= 0, & \hat{L}_+ Y_2(\theta, \varphi) &= 0 \\ \hat{P}_+ f_2(x) &= 0, & \hat{L}_- Y_1(\theta,\varphi) &= 0 \\ \hat{P}_+ g_1(x) &= 0, & \hat{L}_+ \tilde{Y}_2(\theta, \varphi) &= 0 \\ \hat{P}_- g_2(x) &= 0, & \hat{L}_- \tilde{Y}_1(\theta, \varphi) &= 0 \end{aligned} \tag{24}$$

where

$$\hat{P}_\pm = \frac{d^2}{dx^2} + \frac{\omega^2}{x^2} \pm \frac{i\omega}{x^2} - \frac{\lambda^2}{R_0^2} \tag{25}$$

$$\hat{L}_\pm = \frac{\partial^2}{\partial\theta^2} + \mathrm{ctg}\,\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{1}{4}\cos^2\theta \pm i\cos\theta\frac{\partial}{\partial\varphi} + \frac{\partial^2}{\partial\varphi^2} - \frac{1}{2}\right) + \lambda^2 \tag{26}$$

Equation (26) shows that its solution is the spin-1/2 weighted spherical harmonic (Li and Mi, 1999). The separation constant is

$$\lambda = l + \tfrac{1}{2} \tag{27}$$

where $l > \frac{1}{2}$. Using the WKB approximation with $f \sim \exp[iS(x)]$, we find

$$p_x = \partial_x S = \frac{\omega^2}{x^2} - k^2 \tag{28}$$

where $k^2 = \lambda^2/R_0^2$. The number of modes reads

$$n = \frac{1}{\pi}\int_\epsilon^{\omega/k} p_x\,dx = \frac{1}{\pi}\left[-\sqrt{\omega^2 - k^2\epsilon^2} + \omega\ln\frac{k\epsilon}{\omega - \sqrt{\omega^2 - k^2\epsilon^2}}\right] \tag{29}$$

and the number of modes with energy less than ω is given by

$$g(\omega) = \int n(2l + 1)\, dl$$
$$= \frac{2R_0^2}{\pi} \int k\, dk \left[\omega \ln\left(\frac{k\epsilon}{\omega - \sqrt{\omega^2 - k^2\epsilon^2}} \right) - \sqrt{\omega^2 - k^2\epsilon^2} \right] \quad (30)$$

The low limit of intergration is determined by $l > \frac{1}{2}$. The density of states reads

$$\frac{dg(\omega)}{d\omega} = \frac{2R_0^2}{\pi} \int_{1/R_0}^{\omega/\epsilon} k\, dk \ln \frac{k\epsilon}{\omega - \sqrt{\omega^2 - k^2\epsilon^2}}$$
$$= \frac{2R_0^2\omega^2}{\pi\epsilon^2} \int_b^1 y \ln \frac{1 + \sqrt{1 - y^2}}{y}\, dy$$
$$= \frac{2R_0^2\omega^2}{\pi\epsilon^2} \left[\frac{-b^2}{2} \ln(1 + \sqrt{1 - b^2}) + \frac{\sqrt{1 - b^2}}{2} + b^2 \ln b \right] \quad (31)$$

where $y = \epsilon k/\omega$, $b = \epsilon/\omega R_0$. In the previous section, we saw that ϵ is very small, and R_0 is very large for a general black hole with the mass of the sun and length of 3 km, 10^{38} mutiplied by the Planck length. Therefore, when $\epsilon/R_0 \to 0$, $b \to 0$, the density of states

$$\frac{dg(\omega)}{d\omega} \to \frac{R_0^2\omega^2}{\pi\epsilon^2} \quad (32)$$

The free energy reads

$$F = -\frac{1}{\beta} \int_0^\infty dg(\omega) \ln(1 + e^{-\beta\omega})$$
$$= -\frac{R_0^2}{\beta\pi\epsilon^2} \int_0^\infty d\omega\, \omega^2 \ln(1 + e^{-\beta\omega}) = -\frac{7\pi^3 R_0^2}{360\beta^4\epsilon^2} \quad (33)$$

When the particle spin is taken into account, the degree of freedom will contribute a factor to the free energy. For a massless particle with helicity the factor is 1, otherwise it is 2. We have

$$F = -q \frac{7\pi^3 R_0^2}{360\beta^4\epsilon^2}, \qquad q = 1, 2 \quad (34)$$

$$S = \beta^2 \frac{\partial F}{\partial \beta} = q \frac{7R_0^2}{180\epsilon^2} = \frac{7q}{8} \frac{\mathrm{A}}{360\pi\epsilon^2} \quad (35)$$

The entropy of the Dirac field is $7q/8$ multiplied by that of the scalar field if one takes the same cutoff. The difference is caused by two distinct statistical laws and two kinds of spin. Despite this, the entropies of the two fields are both proportional to the area of the horizon, which is the intrinsic and universal property of the event horizon.

5. ON EXTREME BLACK HOLES

In the previous two sections, we have shown that the dynamical behavior of fields near the horizon is responsible for the geometrical character of black-hole entropy. A new method for calculating the black-hole entropy has been proposed, which make it easy and brief to calculate the entropy of high-spin fields in curved space. But the method is invalid for extreme black holes because of their vanishing temperature.

The entropy of an extreme black hole is an intriguing topic. According to Hawking (Hawking and Horowitz, 1995), the entropy of a black hole is related to the nontrival topology of spacetime. The Bekenstein–Hawking entropy is given by

$$S = \frac{\chi}{8} A \tag{36}$$

where χ is the Euler index, $\chi = 2$ for a nonextreme black hole, and $\chi = 0$ for an extreme black hole. However, the entropy of an extreme black hole is still propotional to the area of the horizon according to the string viewpoint (Strominger and Vafa, 1996).

In classical thermodynamics, the third law implies vanishing entropy of a system with zero temperature. However, quantum statistical mechanics allows a constant entropy per particle of a system, and therefore the total entropy of a system may be nonvanishing. Thermodynamics is a classical and phenomenal theory. Correspondingly, the so-called Bekenstein–Hawking entropy is derived from the contribution of a classical action, and formula (36) only has a classical meaning. The string viewpoint on the nonvanishing entropy of an extreme black hole may be reasonable, which is analogous with quantum statistical mechanics.

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